The baryon static potential from lattice QCD

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Lattice QCD simulations offer the possibility of determining the potential between three static quarks from first principles. We review the status of such simulations, and the relative standing of the two theoretical proposals for the baryonic potential: the Delta law (sum of two-body terms) and the Y law (length of three flux strings joined together at a junction). We also present new results on the leading Lüscher-like corrections to the asymptotic linear potential.

1. INTRODUCTION

The potential between 3 static color charges can help predict the masses of baryons made of 3 heavy, non-relativistic quarks. It also reveals by which mechanism the color interaction confines the 3 quarks into a color singlet. In the $q\bar{q}$, mesonic case, the 2 charges are confined by the formation of a string of flux joining them. The string energy grows in proportion to its length, which produces linear confinement. Moreover, the worldsheet spanned by this string fluctuates, and the Casimir energy of such Gaussian fluctuations is expressed by the Dedekind η-function, which generates a $\frac{\pi(d-2)}{24}\frac{1}{r}$ correction to the linear potential in d dimensions, the so-called Lüscher correction [1]. This correction is universal, because it does not depend on the underlying linearly confining theory.

The numerical study of linear confinement in the baryonic case has been the object of old [2] and new [3, 4, 5, 6, 7] lattice simulations. The phenomenological question is: is confinement accompanied by the formation of strings of flux as in the $q\bar{q}$ case? and if so, is the string tension the same? The answer is yes to both questions, and we review the evidence below in Sec. II. When the separation between any 2 quarks is large, 3 strings of color flux form, which meet at a junction. Their worldsheet forms a 3-bladed surface, whose Casimir energy produces a Lüscher-like $\frac{1}{r}$ correction, which depends on the geometry of the qqq triangle but not on the confining theory. This correction can be computed analytically [5], and we compare it with numerical simulations in Sec. III.

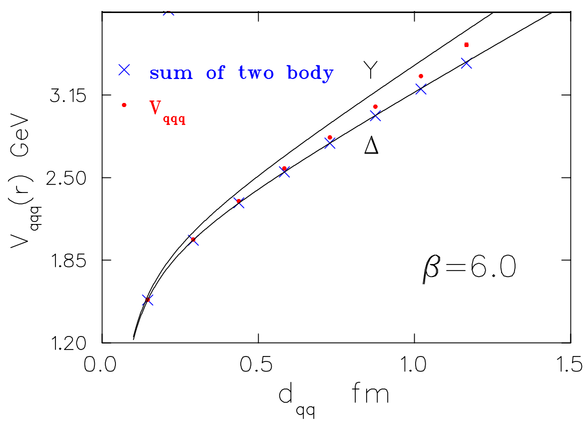

Figure 1. Static baryon potential for equilateral geometry, showing agreement with Δ-law at short distances and departure for $d_{qq} \gtrsim 0.8$ fm. From [4].

β	a(fm)	$\sigma_{q\bar{q}}$	σ_{qqq}	$1 - \frac{\sigma_{qqq}}{\sigma_{q\bar{q}}}$
5.7	0.19	0.160348	0.155624	3%
5.8	0.14	0.108028	0.10316	4.5%
6.0	0.10	0.5007	0.4674	6.6%

Table I. Results from [7], demonstrating departure from Y-law at short distances (high β). At short distances, the effective σ_{qqq} is reduced, indicating that the potential grows more slowly, as expected from the Δ-law.

2. CHANGING FROM Δ-LAW TO Y-LAW WITH INCREASING DISTANCE

At short distance, perturbation theory applies. One-gluon exchange between 2 quarks i and j of a baryon at distance r_{ij} produces an attractive force, which is 1/2 that between q and $\bar{q}$ at the same distance, because of the different color index contractions. Therefore, the baryonic potential at short distances obeys the so-called Delta law:

$$V_{qqq}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \approx \frac{1}{2} \sum_{i<j} V_{q\bar{q}}(r_{ij}) \tag{1}$$

In this case, only 2-body qq forces are at work. If this ansatz persists at large distances, then the potential will grow linearly as $\frac{1}{2}\sigma_{q\bar{q}}(r_{12} + r_{23} + r_{31})$, hence the name Δ-law.

At large distances, if 3 flux strings form, with energy proportional to their length, they will meet at a junction x_J which in the groundstate will localize at the Steiner point which minimizes the total string length $L_Y = \min_{x_J} \sum_{i=1}^{3} |\vec{r}_i - \vec{x}_J|$. The potential will grow as the Y-law:

$$V_{qqq}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \approx \sigma_{q\bar{q}} L_Y \tag{2}$$

Because the difference between L_Y and $\frac{L_\Delta}{2}$ is at most $\sim 15\%$, it takes good numerical accuracy to distinguish between the two ansätze. Moreover, the Δ-law holds at short distances, so the question really is: at which distance does the Y-law take over, if it does? Two groups have addressed this question with the required accuracy.

- Ref.[3] points out that no significant deviation from the Δ-law is observable up to qq separations of about 0.8 fm. In the follow-up Ref.[4], several technical refinements, among them a variational basis of junction locations, allow to uncover clear deviations from the Δ-law, consistent with the Y-law. Fig. 1, taken from Ref.[4], shows the difference between the measured V_{qqq} and the measured Δ-law eq.(1). Note that no

fitting is involved. At distances larger than ~ 0.8 fm, the slope of V_{qqq} (i.e. the baryonic force) becomes consistent with that of the Y-law eq.(2).

• The other group [6, 7] obtains similar numerical results, but analyzes them differently. The data are fitted with a Δ- or Y-ansatz, and the latter works best. In this fit, the baryonic string tension σ_{qqq} is a fitting parameter, allowed to differ from the measured $\sigma_{q\bar{q}}$. A selection of the results of Ref.[7] is shown in Table I, illustrating that at higher β, where the lattice is fine and small distances dominate the fit, σ_{qqq} comes out systematically smaller than $\sigma_{q\bar{q}}$. Equivalently, if one would enforce instead $\sigma_{qqq} = \sigma_{q\bar{q}}$, the effective string length is smaller than L_Y, increasingly so at shorter distances (higher β), as expected from the transition to a Δ-law at short distance. So the results of Ref.[7] are simply inconsistent with their simplifying statement that "the Coulomb plus Y-type linear potential is accurate at the 1% level". The Δ- to Y-law transition is gradual, and occurs around qq separations $\mathcal{O}(0.8$ fm$)$. The string tension $\sigma_{qqq} = \sigma_{q\bar{q}}$ defines the unique correlation length emerging from the gluon dynamics. The dominance of the Y-law occurs for strings of length ~ 0.5 fm or more. At shorter distances, the "string" is as fat as it is long, and the flux picture underlying the Y-law breaks down. This is similar to the $q\bar{q}$ case. However, it indicates that a flux tube description of a baryon, as e.g. in [8], may be a mediocre approximation.

Baryonic flux strings have been exhibited, in quenched and full QCD after Abelian projection [9], and in a gauge-invariant way after smearing [10].

3. BARYONIC LÜSCHER-LIKE CORRECTION

At large distances, where the string description applies, the worldsheet of the three strings describes a 3-bladed surface. Assigning to such a surface an action proportional to its area, one can analytically integrate over small Gaussian fluctuations, much like in the $q\bar{q}$ case. The essential difference is in the boundary conditions: each of the 3 sheets has one fixed (q) and one fluctuating (junction) boundary. At the junction, continuity and balance of forces cause a mixed Dirichlet-Neumann condition [11]. Summation over all eigenenergies results in a Lüscher-like correction to the V_{qqq} potential [5]:

$$C\frac{1}{L_Y} \tag{3}$$

where C depends on the geometry of the qqq triangle and on the space-time dimension.

• In $d = 3$, C is non-negative. Therefore, the linear asymptotic behaviour $V_{qqq} \sim \sigma L_Y$ is approached from *above*, and the potential has an inflexion point as a function of the triangle size (for fixed geometry). For the equilateral case, $C = 0$, due to a cancellation between the 2 types of eigenmodes: all 3 blades vibrating in phase (1 mode), and 2 blades in phase opposition with the third at rest (3 modes, of which 2 are independent), carrying a factor -1/2 due to the different boundary conditions.

• In $d = 4$, fluctuations of the junction outside the qqq plane generate a large negative $\frac{1}{L_Y}$ correction, making C negative always, with less sensitivity to the qqq geometry.

Fig. 2 shows $C \times \frac{24}{\pi}$ in $d = 3$ and 4, as a function of the qqq triangle geometry,

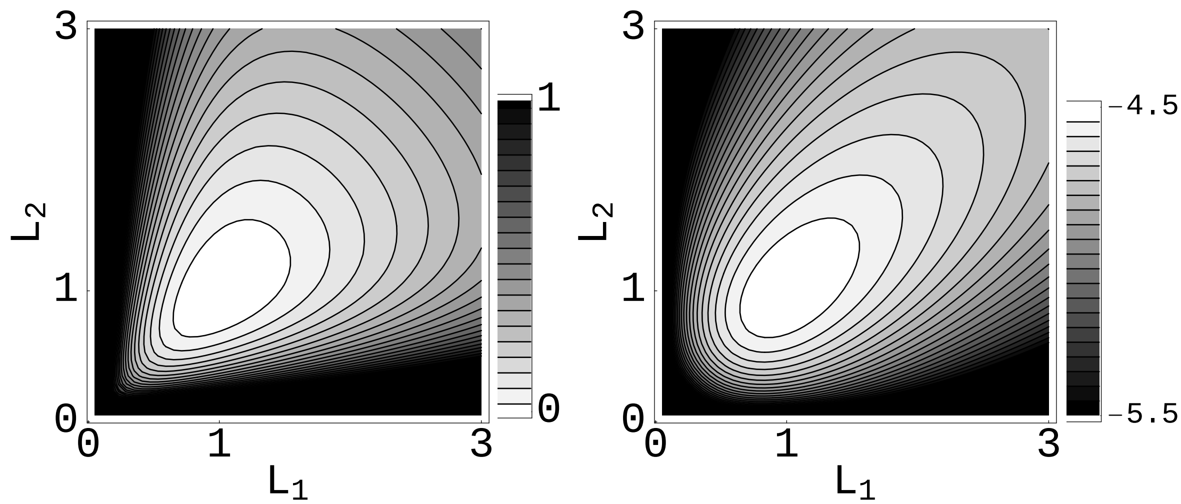

Figure 2. Coefficient of the Lüscher-like term $\frac{\pi}{24}\frac{1}{L_Y}$ as a function of the 3 relative string lengths $(L_1, L_2, L_3 = 1)$, in $d = 3$ (left) and 4 (right). In $d = 3$, $C > 0$, so that V_{qqq} has an inflexion point.

represented by coordinates (L_1, L_2) which are the lengths of 2 blades relative to $L_3 = 1$. The equilateral case is $(1, 1)$, the right-angled isosceles case is $(\sqrt{3}+1, \sqrt{3}+1)$.

These analytic predictions can be confronted to numerical simulations. Since they are universal, we chose to simulate the simplest gauge theory, Z_3 in $d = 3$. To reach the required distances and accuracy, we first perform a duality transformation. A Wilson loop expectation value in the Z_3 gauge theory is equal to the free energy of an interface bounded by this loop, in the dual 3-states Potts model. We measure the variation of this free energy with the area of the interface by using the "snake" algorithm [12], which increases the interface area one plaquette at a time. In fact, only one simulation is required for each elementary displacement of a quark as in [13].

All simulations are performed in the 3-states Potts model at $\beta = 0.60$ on a 48^3 lattice, using Swendsen-Wang updates and a multishell arrangement for variance reduction [14]. The crucial point of the algorithm is that a determination of the baryonic force to a given accuracy requires computer resources *independent* of the qqq separation. This makes the approach extraordinarily efficient.

As an illustration, Fig. 3 shows the mesonic force $-\frac{dV_{q\bar{q}}}{dr}$ for $q\bar{q}$ distances up to ~ 6 fm (the lattice spacing $a \sim 0.18$ fm is obtained by fixing the string tension $\sqrt{\sigma} = 440$ MeV). The corresponding Lüscher term indeed approaches the expected $\frac{\pi}{24}\frac{1}{r^2}$, with a $\frac{1}{r^4}$ subleading correction of the sign predicted in [15], with a large but a-dependent magnitude.

In the baryonic case, we studied 3 qqq geometries: equilateral, right-angled isosceles, and quark-diquark-like (2 quarks fixed, the third at distance h along the mediatrix).

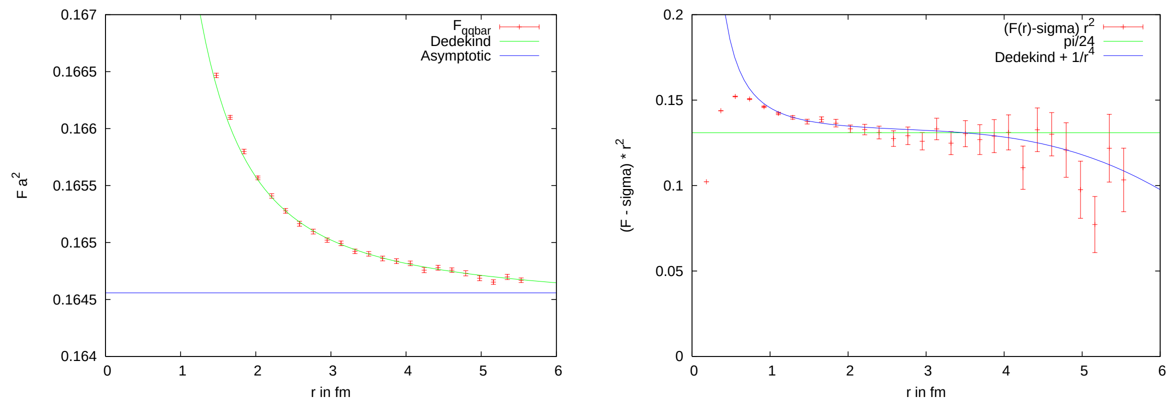

Figure 3. Mesonic ($q\bar{q}$) force in the $3d$ Z_3 gauge theory: force versus distance (left) and coefficient of the Lüscher correction (right). The solid curve corresponds to the Dedekind function at the temperature of our study (hence the bending down), plus a $1/r^4$ term with a fitted coefficient. The overshoot above $\pi/24$ has been predicted [15] but not yet observed in $SU(N)$ simulations. Note the very large distances probed.

In all 3 cases, the asymptotic Y-law is clearly established, with $\sigma_{qqq} = \sigma_{q\bar{q}}$ (compare the force in Fig. 3 (left) and Fig. 4 (left)). But the approach to asymptotia is very geometry-dependent, as seen in Fig. 4 (left). It is fastest in the equilateral case, as predicted since the leading correction vanishes for this geometry ($C = 0$). Indeed, our quasi-equilateral data are well described by a force $(\sigma - c_1/L_Y^2 + c_2/L_Y^4)$, and the fitted value of c_1 is consistent with the string prediction (Fig. 4, middle). This is also true for the right-angled isosceles case, but the subleading correction is very large. For the quark-diquark-like case, the geometry changes as the third quark moves. The string prediction is shown by the solid curve in Fig. 4 (right), together with the mesonic value $\pi/24$ corresponding to the quark-diquark asymptotic limit. The data are completely consistent with the string prediction. Note however the large quark separations necessary to recover good agreement in all cases.

Therefore, although subleading corrections turn out to be sizable, our numerical study confirms quantitatively the analytic calculation of the leading, universal $1/L_Y$ correction to the Y-law.

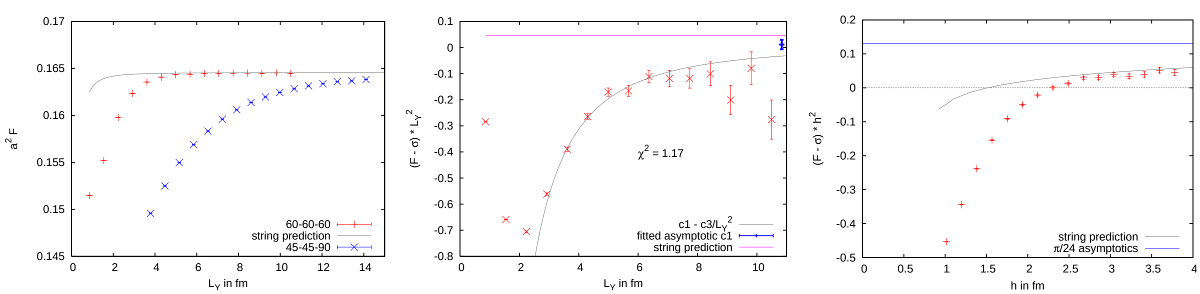

Figure 4. Baryonic (qqq) force and Lüscher term in the $3d$ Z_3 gauge theory. Left: force versus distance for equilateral and 45-45-90 degree geometries. Middle: the coefficient C (equilateral case) of the Lüscher term is consistent with the string prediction $C \approx 0$ after including a subleading $1/r^4$ correction. Right: C (quark-diquark case) is consistent with the string prediction and approaches $\pi/24$ in the diquark limit.

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